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Pytkeev spaces and sequential extensions

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Abstract

In this paper we construct, in response to a question of Malykhin and Tironi, a ZFC example of a perfectly normal Pytkeev space which has no sequential extensions. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

A space X is called:

- (i) sequential if, for every non-closed subset A of X , there exists a sequence $\{x_n\}_n \subset A$ converging to a point of $\overline{A} \setminus A$ (see, e.g., [5,6]);
- (ii) subsequential if it has a sequential extension (see, e.g., [7]);
- (iii) Pytkeev if, for every $A \subset X$ and every $x \in \overline{A} \setminus \{x\}$, there exists a countable family of infinite subsets of A such that every neighbourhood of x contains an element of this family [10].

It turns out that a space which has a sequential T_1 -extension is Pytkeev [10,12].

Moreover every Pytkeev space X has countable tightness, i.e., if $A \subset X$ and $x \in \overline{A}$, then there is a countable subset B of A such that $x \in \overline{B}$ (see [10] for a detailed study of the relationships between these and other related properties).

In [10] it is shown that:

- (i) Every compact Hausdorff space having countable tightness is a Pytkeev space;

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- (ii) Under the Jensen's Principle \diamond the Ostaszewski compact space [11] is an example of a Pytkeev space which is not subsequential.

Since under Proper Forcing Axiom every compact Hausdorff space of countable tightness is sequential ([1], see also [2,3]), the following (ZFC) problem posed by Malykhin and Tironi in [10] naturally arises:

Problem. Does there exist a Pytkeev T_1 -space which has no sequential T_1 -extensions?

In this paper we give a positive answer to the above problem.

We refer the reader to [5,8,9] for notations and terminology not explicitly given.

1. The counterexample

Example. A Pytkeev T_6 -space which is not subsequential.

Let $X = (\omega \times \omega) \cup \{p\}$ (where $p \notin \omega \times \omega$) and let \mathcal{U} be a free ultrafilter on ω . If $F \subset G \subset \omega$ and $\phi: G \rightarrow \omega$ define $V_{F,\phi} = \{(n, i): n \in F, i > \phi(n)\} \cup \{p\}$.

Let τ be the topology on X in which every point of $\omega \times \omega$ is open and basic neighbourhoods of p are of the form $V_{F,\phi}$, where $F \in \mathcal{U}$ and $\phi \in \omega^G$ with $F \subset G \subset \omega$.

(X, τ) is a countable T_1 -space with only one non-isolated point, so it is a perfectly normal space.

Claim 1. (X, τ) is a Pytkeev space.

Proof. It is enough to show the Pytkeev property for the couple (p, A) , where $A \subset X$ and $p \in \overline{A} \setminus A$.

Set $H = \{n \in \omega: |A \cap (\{n\} \times \omega)| = \omega\}$.

Let us show that $H \in \mathcal{U}$. Assume not, then $L = \omega \setminus H \in \mathcal{U}$. Let $\psi: L \rightarrow \omega$ be defined by $\psi(n) = \max\{i: (n, i) \in A\}$, then $V_{L,\psi} \cap A = \emptyset$, a contradiction. So $H \in \mathcal{U}$.

Set $A_{n,m} = \{(n, i) \in A: i > m\}$ for every $n \in H$ and $m \in \omega$. Take a basic neighbourhood $V_{F,\phi}$ of p in X and pick some $n \in F \cap H$, then $A_{n,\phi(n)} \subset V_{F,\phi}$.

Therefore $\{A_{n,m}: n \in H, m \in \omega\}$ is a countable family of infinite subsets of A with the desired property, so X is a Pytkeev space. \square

Claim 2. (X, τ) has no sequential extensions.

Proof. It is enough to show that there is no dense extension (S, σ) of (X, τ) with the following property: for some $\alpha \in \omega_1$ there is an increasing family $\{A_\beta: \beta \leq \alpha + 1\}$ of subsets of S such that $A_0 = A = \omega \times \omega$, $p \in A_{\alpha+1} = S$ and A_β is the sequential closure of order β of A in (S, σ) for every $\beta \leq \alpha + 1$.

Assume not and let α be the smallest countable ordinal for which such a family $\{A_\beta: \beta \leq \alpha + 1\}$ does exist.

Choose a sequence $q_n \in A_\alpha \setminus A$ converging to p (for the sake of convenience, we make the implicit assumption that, when needed, the sequence q_n can be replaced by one of its subsequences). Clearly we can reduce ourselves to the following two cases:

Case 1. q_n and p can be separated by disjoint open subsets of S for every n .

Case 2. $q_n \in \overline{V}$ for every neighbourhood V of p in S .

Proof of Case 1. Clearly we may assume that, for every $n \in \omega$, there is a pair of disjoint open sets C_n, D_n of S such that:

- (i) $q_n \in C_n$,
- (ii) $p \in D_n$,
- (iii) $C_{n+1} \cup D_{n+1} \subset D_n$,
- (iv) $D_n \cap X = V_{F_n, \phi_n}$, where $\{F_n: n \in \omega\}$ is a strictly decreasing sequence of members of \mathcal{U} .

Since \mathcal{U} is an ultrafilter, we may assume that $F = \bigcup \{F_{2n} \setminus F_{2n+1}: n \in \omega\} \in \mathcal{U}$. Let $\phi: F \rightarrow \omega$ be the mapping defined by $\phi(j) = \phi_{2n}(j)$ whenever $j \in F_{2n} \setminus F_{2n+1}$.

We claim that $q_{2n} \notin \overline{V}_{F, \phi}$. Since C_{2n} is a neighbourhood of q_{2n} , it is enough to show that $C_{2n} \cap V_{F, \phi} = \emptyset$. Suppose not, then there is some $(k, i) \in C_{2n}$ such that $k \in F$ and $i > \phi(k)$. Let $n \in \omega$ be such that $k \in F_{2n} \setminus F_{2n+1}$, then $\phi(k) = \phi_{2n}(k) > i$, so $(k, i) \in V_{F_{2n}, \phi_{2n}} \subset D_{2n}$, hence $(k, i) \in C_{2n} \cap D_{2n}$, a contradiction.

Therefore $q_{2n} \notin \overline{V}_{F, \phi}$. Since $\overline{V}_{F, \phi}$ is a (closed) neighbourhood of p in (S, σ) , it follows that $\{q_n\}_n$ cannot converge to p .

Proof of Case 2. Let $\mathcal{W}(x)$ be the family of all neighbourhoods of x in S and let $\mathcal{U}_x = \{V \cap A: V \in \mathcal{W}(x)\}$. Let us show that there are no points $q \in A_\alpha \setminus A$ such that $\mathcal{U}_p \subset \mathcal{U}_q$. Suppose not, and take some $q \in A_\alpha \setminus A$ such that $\mathcal{U}_p \subset \mathcal{U}_q$.

Let $\gamma \in \alpha$ be such that $q \in A_{\gamma+1} \setminus A_\gamma$, and set $Y = A_\gamma \cup \{q\}$.

Let $\rho = \{V \cap A_\gamma: V \in \sigma\} \cup \{V \cap Y: q \in V \in \sigma, V \cap A \in \mathcal{U}_p\}$. Clearly ρ is a topology on Y such that $\rho|_{A_\gamma} = \sigma|_{A_\gamma}$. Therefore A_β is contained in the sequential closure of order β of A in (Y, ρ) for every $\beta \leq \gamma$.

Now let us take a sequence $\{x_n: n \in \omega\} \subset A_\gamma$ converging to q in (Y, σ) . Clearly $\{x_n: n \in \omega\}$ converges to q in (Y, ρ) too.

Hence Y is the sequential closure of order $\gamma + 1$ of A in (Y, ρ) . Since the subspace $A \cup \{q\}$ of (Y, ρ) is homeomorphic to the subspace $A \cup \{p\} = X$ of (S, σ) , we reach, by the minimality of α , a contradiction.

Therefore $\mathcal{U}_p \not\subset \mathcal{U}_q$ for every $q \in A_\alpha \setminus A$.

Now it suffices to consider the following two subcases:

Subcase 1. $q_n \notin \overline{H \times \omega}$ for every $H \notin \mathcal{U}$ and $n \in \omega$.

Since $\mathcal{U}_p \not\subset \mathcal{U}_{q_n}$, there is, for every n , some $B_n = V_{F_n, \phi_n} \setminus \{p\} \in \mathcal{U}_p \setminus \mathcal{U}_{q_n}$, with $\phi_n \in \omega^\omega$. Clearly we may assume $\phi_n < \phi_{n+1}$ (i.e., $\phi_n(i) \leq \phi_{n+1}(i)$ for every $i \in \omega$).

By hypothesis there is a neighbourhood V of q_n such that $V \cap A \subset F_n \times \omega$. Since $B_n \notin \mathcal{U}_{q_n}$ and \mathcal{U}_{q_n} is a filter on A , it follows that $W \cap A \not\subset B_n$ for every neighbourhood W of q_n . Therefore $q_n \in \overline{V_{F_n, \neg \phi_n}}$, where $V_{F, \neg \phi} = (F \times \omega) \setminus V_{F, \phi}$ for every $F \in \mathcal{U}$ and $\phi \in \omega^\omega$.

Since $q_n \notin \overline{(\omega \setminus F) \times \omega}$ for every $F \in \mathcal{U}$ and $n \in \omega$, it follows that $q_n \in \overline{V_F, \neg \phi_n}$ for every $F \in \mathcal{U}$.

Clearly we may take a strictly decreasing sequence $\{C_n\}$ of elements of \mathcal{U} having the following property: if G_n is a neighbourhood of p such that $G_n \cap A = V_{C_n, \phi_n}$, then $q_i \in G_n$ for every $i \geq n + 1$.

Set $F = C_1$ and let ϕ be defined by $\phi(i) = \phi_n(i)$ whenever $i \in C_n \setminus C_{n+1}$.

Let G be an open set such that $G \cap A = V_{F, \phi}$, then $q_n \notin G$. In fact $q_n \in \overline{V_{F_n, \neg \phi_n}}$ and $G \cap V_{F_n, \neg \phi_n} \subset V_{F, \phi} \cap V_{F_n, \neg \phi_n} = \emptyset$ (observe that $\phi_n|_{F_n} < \phi|_{F_n}$). Since $\{q_n: n \in \omega\}$ converges to p , we reach a contradiction.

Subcase 2. For every $n \in \omega$ there is some $H_n \notin \mathcal{U}$ such that $q_n \in \overline{H_n \times \omega}$.

We claim that the sequence $\{H_n: n \in \omega\}$ can be taken in such a way that $H_n \cap H_m = \emptyset$ whenever $n \neq m$.

If this were not the case, we may assume that there is some $\kappa \in \omega$ such that for every $n \leq \kappa$ there is $H_n \notin \mathcal{U}$ with $q_n \in \overline{H_n \times \omega}$ and $q_i \notin \overline{H \times \omega}$ for every $i \geq \kappa$ and every $H \notin \mathcal{U}$ such that $H \subset \omega \setminus (\bigcup_{n \leq \kappa} H_n)$.

At this point we can apply Subcase 1 replacing $\{q_n: n \in \omega\}$ by $\{q_n: n > \kappa\}$ and A by $[\omega \setminus (\bigcup_{n \leq \kappa} H_n)] \times \omega$.

So let $\{H_n: n \in \omega\}$ be a pairwise disjoint family such that for every $n \in \omega$, $H_n \notin \mathcal{U}$ and $q_n \in \overline{H_n \times \omega}$.

Now let $U \in \mathcal{U}$ such that $U \cap (\bigcup \{H_{2n}: n \in \omega\}) = \emptyset$ (clearly we may assume that $\bigcup \{H_{2n}: n \in \omega\} \notin \mathcal{U}$) and let V be a neighbourhood of p in S such that $V \cap A = U \times \omega$. Since every neighbourhood of q_{2n} meets $H_{2n} \times \omega$ (which is disjoint from $U \times \omega$), it follows that $q_{2n} \notin V$ for every n , a contradiction. \square

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